International Journal of Theoretical Physics, Vol. 46, No. 5, May 2007 (© 2006)
DOI: 10.1007/s10773-006-9269-y

Energy Transfer in Spectra of the *d*-Dimensional Past Grid Turbulence

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Received 10 July 2006; accepted: 29 September 2006
Published Online: November 29 2006

Free decay theory of the homogeneous and isotropic developed turbulence is considered in the d-dimensional case. The basic quantities under our consideration are the kinetic energy spectrum $E(k, t)$ and energy transfer spectrum $T(k, t)$ as functions of wave number k and decay time t. Starting point for studying E and T represents their adaptation from the stationary model which predicts the Kolmogorov spectrum which is multiplicatively dependent on an unknown scaling function F. In order to study the spectra of decaying turbulence both parameters l and ε are supposed to be dependent on t. Formerly derived basic integro–differential equation for F (by Adzhemyan *et al.*, 1998) has been here solved numerically in the dimension interval $d \in (2, 3)$ for two cases of the Saffman invariant and the Loitsyansky integral fixing an arbitrary theory parameter α ($\alpha = 2$ and 4, correspondingly). The energy transfer spectrum $T(k)$ has been analyzed for several dimensions $d \leq 3$ showing the presence of integration regions in the wavenumber space where an inverse energy cascade can occur.

KEY WORDS: decay turbulence; spectral energy transfer; renormalization group.
PACS numbers: 47.27.ef, 47.11.-j, 47.27.er

1. INTRODUCTION

The Kolmogorov's universality hypothesis (Kolmogorov, 1941) which supposes that inertial range statistics of strongly developed hydrodynamic (HD) turbulence is independent of the geometry of boundary conditions and dynamics of dissipation scales, has received a remarkable experimental support [see in Frisch (1995) and ref. therein]. In order to establish the validity of the phenomenologically obtained $-5/3$ law of the kinetic energy spectrum, first the renormalization group (RG) method was applied (Dominicis and Martin, 1979) in the framework of the Wyld's statistical model of the randomly stirred fluid (Wyld,

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1961). Later developments of the different RG variants in the stochastic hydrodynamics (Yakhot and Orszag, 1986a, b; Dannevik *et al.*, 1987; Adzhemyan *et al.*, 1989a, b) were stimulated by the effort to calculate the Kolmogorov constant.

It is a widely spread opinion (Monin and Yaglom, 1975) that spectral properties of the energy–containing scales are non universal, i.e. the lowest bound of spectrum evolves due to anisotropy and finite–size effects. The issue of the universality of energy–containing range has been raised by George (1992), who analyzed the spectra measured at intermediate distances behind the stirring grid (i.e. at intermediate decay stages), where such integral quantities like the kinetic energy or dissipation rate satisfy the power laws in time. By indicating the presence of a specific scaling or rather variety of scaling forms in decay statistics, George's idea extends the standard picture of universality. This idea has been developed in the previous paper (Adzhemyan *et al.*, 1998) where a more quantitative scaling theory of decay was formulated. The purpose of the presented paper is to develop this scaling theory in general d-dimensional space as well as to understand the energy transfer along k-spectrum in dependence on d.

It is well known (Olla, 1991, 1994; Honkonen and Nalimov, 1996) that for certain critical dimension $d_c \in (2, 3)$ in d-dimensional HD turbulence the spectrum becomes to be non-stable – the scaling regime does not exist. Here we try to find out how the decaying spectrum depends on the space dimension d.

The present paper follows the plan: in Section 2 we review the known equations of the past grid turbulence. Section 3 summarizes the quantum–field RG results obtained for the energy spectrum and energy transfer of the inertial and energy–containing range which represent the starting point of decay analysis. In Section 4 the d-dimensional model of the decay is introduced and integro–differential equation for the scaling function is derived. Method of the scaling function calculation with the special emphasis on the various asymptotics of the spectra is also discussed. Numerical computing of the scaling function in Section 5. is suggested for d-dimensional decay with the Saffman's invariant (part 5.1.) as well as the Loitsyansky invariant (part 5.2.). An analysis of the energy transfer is given in Section 6.

2. STOCHASTIC MODEL OF STATIONARY ISOTROPIC TURBULENCE

The basic equation which reflects the statistics of the strong turbulence is the randomly forced Navier–Stokes equation

$$\partial_\tau v_j = \nu_0 \nabla^2 v_j - \sum_s^d \mathcal{P}_{js}(\nabla)(\mathbf{v}\cdot\nabla)\,\mathbf{v_s} + \mathbf{f_j}, \qquad \nabla\cdot\mathbf{v} = \nabla\cdot\mathbf{f} = \mathbf{0}, \quad (1)$$

where $\mathcal{P}_{js}(\mathbf{k}) = \delta_{js} - k_j k_s/k^2$ is the transverse projection operator which stands to ensure the incompressibility of the fluid. Following the tradition of stochastic models of the turbulence we assume that statistics of the external random force $\mathbf{f}(\mathbf{x}, \tau)$ is isotropic and gaussian. It is completely determined by the statistical average over the ensemble of velocity fluctuations

$$\langle f_j(\mathbf{x}, \tau)\rangle = 0, \qquad \langle f_j(\mathbf{x}_1, \tau_1) f_s(\mathbf{x}_2, \tau_2)\rangle = \delta(\tau_{12})\, \mathcal{P}_{js}(\nabla_{\mathbf{x}_{12}})\, \mathcal{F}(|\,\mathbf{x}_{12}\,|) \tag{2}$$

where $\tau_{12} = \tau_1 - \tau_2$; $\mathbf{x}_{12} = \mathbf{x}_1 - \mathbf{x}_2$ and

$$\mathcal{F}(|\,\mathbf{x}\,|) = \int \frac{d^d\mathbf{k}}{(2\pi)^d}\, e^{i\,\mathbf{k}\cdot\mathbf{x}}\, D^F(k). \tag{3}$$

The forcing spectrum $D^F(k)$ is defined by

$$D^F(k) \equiv D^F(|\,\mathbf{k}\,|) = \overline{D}\, F\,(kl)\; k^{-d}, \tag{4}$$

where $\overline{D}$ is the amplitude of the forcing correlations proportional to the mean injection rate of energy, l is the length scale of size from the energy–containing range. This scale can be associated with the Karman length scale. We assume that the region of those k which are much greater than $1/l$ corresponds to the upper bound of the inertial range. The explicit form of the function $F(kl)$, which is one of our main interests is not specified yet and forcing extension caused by introduction of $F(kl)$ requires some additional remarks. Single point correlation function $\langle \mathbf{f}(\mathbf{x}, \tau_1)\, \mathbf{f}(\mathbf{x}, \tau_2)\rangle$ is proportional to the energy injection rate $\mathcal{F}(0) \propto \int_0^\infty dk\, k^{d-1}\, D^F(k)$. The function $k^{d-1} D^F(k)$ represents the contributions of the separate velocity modes into full energy supply of the forcing. For the scales which are comparable with l the function $F(kl)$ reflects details of the forcing mechanism. In spite of this ambiguity, it is reasonable to expect that maximum of $k^{d-1}\, D^F(k) = \overline{D} F(kl)/k$ should be located at $k \propto 1/l$. The forcing correlator Eq. (4) can be regarded as an improved version of the basic k^{-d} power form of Yakhot and Orszag (1986a, b) and Forster *et al.* (1997) or its extension $k^4\,[\,k^2 + (1/l)^2\,]^{-2} \times k^{-d}$ introduced by Dominicis and Martin (1979) and Adzhemyan *et al.* (1989a, b). The term $1/l$ from the last expression plays the role of the infrared mass (speaking in terminology of the quantum field theory). In the works mentioned, the use of the simplest k^{-d} forcing ($F = 1$) was motivated by the intention to apply the RG method indicating the universality of the inertial range statistics. The additional limitation to the form of $F(kl)$ is the assumption that within the inertial range ($kl \gg 1$) the spectrum $D^F(k)$ approaches k^{-d} tail. This is guaranteed by the normalization $F(\infty) = 1$.

The set of the velocity correlation functions is completely determined by the stochastic model represented by Eqs. (1), (3) and (4). Naturally, the information implemented by $F(kl)$ is consequently reflected by the properties of the correlation functions. Pure inertial range statistics is generated in the special case $F(kl) = 1$.

Let us proceed to the equation of spectral transfer of the energy which plays a central role in description of the strongly developed turbulence. If isotropy of the statistics, incompressibility of the fluid and absence of the external energy sources are supposed, the energy current conservation in the wave–number space is expressed by the equation

$$\partial_t\, E(k,t) = T(k,t) - 2\,\nu_0\, k^2\, E(k,t), \tag{5}$$

where

$$E(k,t) = \frac{1}{2}\, k^{d-1}\, S_d \int \frac{d^d\mathbf{x}_1}{(2\pi)^d}\; e^{-i\mathbf{k}\cdot\mathbf{x}_{12}}\, \langle\, \mathbf{v}(\mathbf{x}_1,t)\,\cdot\,\mathbf{v}(\mathbf{x}_2,t)\,\rangle, \tag{6}$$

$$T(k,t) = -k^{d-1}\, S_d \int \frac{d^d\mathbf{x}_1}{(2\pi)^d}\; e^{-i\mathbf{k}\cdot\mathbf{x}_{12}}\, \langle\, \mathbf{v}(\mathbf{x}_1,t)\,(\mathbf{v}(\mathbf{x}_2,t)\cdot\nabla_{\mathbf{x}_2})\mathbf{v}(\mathbf{x}_2,\mathbf{t})\,\rangle, \tag{7}$$

are the energy spectrum and energy transfer, respectively; $S_d = 2\,\pi^{d/2}/\,\Gamma(d/2)$ is the surface area of d-dimensional sphere of unit radius; $\Gamma(x)$ is the Gamma function; ν_0 is the kinematic viscosity. One can easily verify that energy transfer fulfills the integral identity

$$\int_0^\infty dk\, T(k,t) = 0 \tag{8}$$

which allows to write the equation of the total energy conservation

$$\partial_t\, \mathcal{E} = -\varepsilon. \tag{9}$$

Here the mean kinetic energy $\mathcal{E}(t)$ and the mean energy dissipation rate $\varepsilon(t)$ are defined by the integrals

$$\mathcal{E}(t) = \int_0^\infty dk\; E(k,t), \qquad \varepsilon(t) = 2\nu_0 \int_0^\infty dk\, k^2\, E(k,t). \tag{10}$$

It is well known that Eqs. (5), (9) and (10) are transferable to the experimentally most easily accessible case of the past grid turbulence if the invoking of the Taylor's concept of the frozen turbulence (Taylor, 1938; Monin and Yaglom, 1975) is well justified. For the statistically homogeneous past grid flow, the time of decay t can be simply related to distance measured in a streamwise direction from the position of the stirring grid.

3. CALCULATION OF THE ENERGY SPECTRUM AND ENERGY TRANSFER

To solve the complicated problem of the calculation of the spectral characteristics $E(k)$ and $T(k)$, advanced theoretical tools utilizing the field–theoretic RG (Dominicis and Martin, 1979; Adzhemyan *et al.*, 1983) and *short distance* *expansion* techniques (Adzhemyan *et al.*, 1995, 1999) have been applied. The application was inspired by general approaches developed in the quantum field theory and critical phenomena. Their excellent summary represents the book of Zinn-Justin (1989).

By Dominicis and Martin (1979) and Adzhemyan *et al.* (1983, 1989a, b) was shown that energy spectrum induced by the $\overline{D}\, k^{-d}$ forcing is of the form

$$E(k) = c_E\, (\overline{D})^{2/3}\, k^{-5/3}, \tag{11}$$

where

$$c_E = \frac{(d-1)\,(g_*)^{1/3}\, S_d}{4(2\pi)^d}, \qquad g_* = \frac{16\,(d+2)\,(2\pi)^d}{3(d-1)\, S_d}. \tag{12}$$

Here the dimensionless parameter g_* was fixed by the RG transformation. The presence of the extended forcing (4) is reflected by the energy spectrum

$$E(k) = c_E\, (\overline{D})^{2/3}\, F\,(kl)\; k^{-5/3} \tag{13}$$

of the inertial and energy–containing scales. The only difference between (11) and (13) is the presence of F. Already in the simplest case of the inertial range ($F = 1$), problems appear in the analysis of the triple velocity correlation function (see (7)). This function is related to the energy transfer via an integral formula. In Adzhemyan *et al.* (1998) the reader can find details of its derivation for the extended forcing.

According to Adzhemyan *et al.* (1995) the resulting self–similar form of $T(k)$ is

$$T(k) = c_T\, \overline{D}\, k^{-1}\, \psi^{\{F\}}\,(kl)\,, \qquad c_T = \frac{g_*\, S_d}{2(2\pi)^d}, \tag{14}$$

where the structure of the dimensionless scaling function $\psi^{\{F\}}(\chi)$ of dimensionless variable $\chi = k\,l$ is

$$\psi^{\{F\}}\,(\chi) = c_T' \int_{\triangle} dq\, dp\;\; R(q, p\,; \chi), \qquad c_T' = \frac{S_{d-1}}{2^{d+1}\,(2\pi)^d}, \tag{15}$$

$$\begin{aligned} R(q, p\,; \chi) = K(p, q)\big\{ & F\,(p\chi)\, F\,(q\chi)\,(pq)^{-d-2/3}\,[Q(p, q) + Q(q, p)] \\ & - F\,(\chi)\,\big[F\,(p\chi)\; p^{-d-2/3} Q(p, q) \,+\, F\,(q\chi)\; q^{-d-2/3} Q(q, p)\big]\big\}, \end{aligned} \tag{16}$$

$$K(p, q) = \frac{\left[2(q^2 + q^2 p^2 + p^2) - (1 + q^4 + p^4)\right]^{\frac{d-1}{2}}}{p\,q\,\left(1 + q^{2/3} + p^{2/3}\right)}, \tag{17}$$

$$Q(p, q) = p^4 - p^2 + (d - 1 - p^2)\, q^2. \tag{18}$$

Note that the symbol $\triangle$ in the integral (15) denotes the domain of integration:

$$\triangle \equiv \{(q, p); q \geq 0, |1 - q| \leq p \leq 1 + q\}, \tag{19}$$

and this integral is rewritten to dimensionless variables

$$\left\{p \to p\frac{\chi}{l},\ q \to q\frac{\chi}{l}\right\} \tag{20}$$

which are introduced in analogy with the dimensionless variable $\chi = k\,l$.

We emphasize that inertial range critical exponents $-\frac{5}{3}$ for $E(k)$ [and -1 for $T(k)$] have been calculated perturbatively exactly (Adzhemyan *et al.*, 1983, 1999) using the epsilon–expansion technique (see e.g. Zinn-Justin, 1989). The stationary form of $T(k)$ as given by Eqs. (14)–(18) is consistent with the eddy damped quasi-normal Markovian model (Orszag, 1977; Fournier and Frisch, 1979; Dannevik *et al.*, 1987). The principal difference between the form of the energy transfer resulting from the markovianization and similar form given by Eqs. (14)–(18) consists in the F presence and presence of the constant c_T fixed by the RG flow.

4. BASIC EQUATION FOR THE PAST GRID TURBULENCE DECAY

The decay of the free evolving past grid turbulence can be modeled by an assumption that role of the energy input is played by the $\partial_t E$ term, see justification in (Adzhemyan *et al.*, 1998). Since we are looking for $E(k, t)$ and $T(k, t)$ in the restricted region of the inertial and energy–containing scales, the viscous effects can be neglected and equation of the spectral budget is applicable in the truncated form $\partial_t E = T$.

Assuming the slowness of the time variations and also persistence of the statistics of the spatial turbulent structures, the decay spectra can be constructed by modifying their stationary forms [see Eqs. (13) and (14)]:

$$E(k, t) = c_E\, (\overline{D}(t))^{2/3}\, F\,(kl(t))\; k^{-5/3}, \qquad T(k, t) = c_T\, \overline{D}(t)\, k^{-1}\, \psi^{\{F\}}\,(kl(t))\,. \tag{21}$$

Regardless of the detailed structure of the $\psi^{\{F\}}(\chi)$, Eq. (21) is compatible with the self–similar relations postulated by Karman and Howarth (1938).

If the time–dependent scaling forms obtained from Eqs. (13) and (14) are inserted into inviscid variant of Eq. (5), the equation for the scaling function $F(\chi)$ can be obtained:

$$\phi_1(t)\ \chi^{-2/3}\left[\,1 + \phi_2(t)\ \chi\,\frac{d}{d\chi}\,\right] F(\chi) = \psi^{\{F\}}\,(\chi)\,, \tag{22}$$

where

$$\phi_1(t) = \frac{2\,c_E}{3\,c_T}\,\left[\,\overline{D}(t)\,\right]^{-1/3}\,[\,l(t)\,]^{2/3}\ \partial_t \ln \overline{D}(t), \tag{23}$$

$$\phi_2(t) = \frac{3}{2}\,\frac{\partial_t \ln l(t)}{\partial_t \ln \overline{D}(t)}. \tag{24}$$

The differentiation of Eq. (22) with respect to t gives rise to the auxiliary equation which allows to determine the conditions of its solubility. The complete analysis of Eqs. (15) and (22) shows that most informative physical solution is obtained when both c_1 and c_2 are some integration constants:

$$\phi_1(t) = c_1, \qquad \phi_2(t) = c_2 \tag{25}$$

and

$$\hat{L}\,F(\chi) = \psi^{\{F\}}(\chi)\,, \quad \hat{L} \equiv c_1\,\chi^{-2/3}\left[1 + c_2\,\chi\,\frac{d}{d\chi}\right]. \tag{26}$$

From the normalization $F(\infty) = 1$ and Eq. (26) we conclude that $\hat{L}F|_{F=1}$ asymptotically vanishes

$$\hat{L}F|_{F=1} = c_1\ \chi^{-2/3}, \quad \text{for} \quad c_1 \neq 0, \qquad \chi \gg 1, \tag{27}$$

whereas $\psi^{\{F\}}$ is equal to zero exactly (McComb, 1990),

$$\psi^{\{F\}}(\chi)| = 0 \quad \text{for} \quad F = 1, \tag{28}$$

i.e. when the energy spectrum acquires the Kolmogorov form. This finding is consistent with the weakened variant of Eq. (26)

$$\lim_{\chi\to\infty} (\hat{L}\ F(\chi) - \psi^{\{F\}}(\chi)) = 0. \tag{29}$$

More restrictive condition

$$\lim_{\chi\to\infty}\ \chi^{2/3}\,(\hat{L}\ F(\chi) - \psi^{\{F\}}(\chi)) = 0 \tag{30}$$

is used in Section 4.2. to control the asymptotic behaviour of $\psi^{\{F\}}(\chi)$ and then to leading order asymptotic of $F(\chi)$.

The solutions $\overline{D}(t)$ and $l(t)$ have been obtained from Eq. (25) by Adzhemyan *et al.* (1998):

$$\overline{D}(t) = D_0\,t^{\alpha_D}, \qquad l(t) = l_0\,t^{\alpha_l}. \tag{31}$$

They are compatible with assumption of Karman and Howarth (1938) and it leads to the following decay laws of the kinetic energy and the energy dissipation:

$$\mathcal{E}(t) = \mathcal{E}_0\,t^{\alpha_\mathcal{E}}, \qquad \alpha_\mathcal{E} = \frac{2(\alpha_D + \alpha_l)}{3}, \tag{32}$$

$$\mathcal{E}_0 = \frac{4\,c_E^3\,\alpha_D^2\,l_0^2}{9\,c_T^2\,c_1^2}\,I^{\{F\}}, \tag{33}$$

$$\varepsilon(t) = -\alpha_{\mathcal{E}}\, \mathcal{E}_0\, t^{\alpha_{\mathcal{E}}-1}, \tag{34}$$

$$I^{\{F\}} \equiv \int_0^\infty dx\;\; x^{-5/3}\, F(x). \tag{35}$$

4.1. The Large Scale Structure of the Turbulence

In analogy with McComb (1990) the assumption about the large scale structure of the turbulence has been supplemented,

$$\lim_{k\to 0} \frac{E(k,t)}{k^\alpha} = \Lambda_\alpha > 0, \quad \alpha > 0, \tag{36}$$

where Λ_α and α are some new constant parameters. The choice of α implies the selection of the flow invariant Λ_α [see below]. From Eqs. (13) and (36) it follows that

$$F(\chi) = c_F\, \chi^{(3\alpha+5)/3}, \quad \text{as} \quad \chi \ll 1, \tag{37}$$

$$\Lambda_\alpha = c_F\, c_E\, \overline{D}^{2/3}\, l^{(3\alpha+5)/3}, \tag{38}$$

where c_F is some constant. In the result we obtain

$$c_2 = -\frac{3}{3\alpha+5}, \qquad \alpha_l = \frac{2}{\alpha+3}, \qquad \alpha_D = -\frac{3\alpha+5}{\alpha+3}, \tag{39}$$

and $\alpha_{\mathcal{E}}$ is defined by relation (32).

The normalization $F(\infty) = 1$ implies that the inertial range spectrum $C_k\, \varepsilon^{2/3}\, k^{-5/3}$ is achieved when $k \gg 1/l$. Therefore, in analogy with the stationary case, the Kolmogorov constant is defined here by the relation

$$C_k = \frac{E(k,t)}{(\varepsilon(t))^{2/3}\, k^{-5/3}\, F(kl)}. \tag{40}$$

The Eqs. (13) and (40) involve two different representations of the same energy spectrum. The time independence of C_k stems from Eq. (40) joining $\alpha_{\mathcal{E}}$ and α_D.

Let us remark that from Eqs. (10) and (40) one has

$$\mathcal{E} = C_k\, I^{\{F\}}\, \varepsilon^{2/3}\, l^{2/3}, \tag{41}$$

and in the result

$$I^{\{F\}} = \frac{d}{2\, C_k}. \tag{42}$$

Note in addition to Eqs. (25) that the connection between the constant c_1 and C_k is (Adzhemyan *et al.*, 1998):

$$c_1 = -\frac{c_{10}}{\sqrt{C_k}}, \quad \text{where} \quad c_{10} = \frac{2\,(3\,\alpha + 5)\,(c_E)^{3/2}}{3\,d\,(\alpha + 1)\,c_T}. \tag{43}$$

Let us now return to the large scale limit (36). One can convince that the value of Λ_α is connected with the longitudinal velocity pair correlation function

$$B_{LL}(|\mathbf{x}_{12}|, t) = \sum_{j,s}^{d} \langle v_j(\mathbf{x}_1, t)\, v_s(\mathbf{x}_2, t)\rangle\, \frac{(\mathbf{x}_{12})_j\,(\mathbf{x}_{12})_s}{|\mathbf{x}_{12}|^2}, \tag{44}$$

which can be related to $E(k, t)$ spectrum by means of the relation

$$B_{LL}(x, t) = \int_0^\infty dk\, E(k, t)\, \rho_L(kx), \qquad \rho_L(y) = \frac{2\,(\sin y - y \cos y)}{y^3}. \tag{45}$$

This correlation function determines some important characteristics of the phenomenology of decay, namely, dimensional integrals (Monin and Yaglom, 1975)

$$\mathcal{I}_\alpha = \int_0^\infty dx\, x^\alpha\, B_{LL}(x, t). \tag{46}$$

In fact, using (40) and (38) one obtains the relations

$$\mathcal{I}_\alpha = \frac{\Lambda_\alpha}{c_F} \int_0^\infty dy\, y^\alpha\, \beta_L(y), \qquad \beta_L(y) = \int_0^\infty d\chi\, \rho_L(\chi\, y)\, \chi^{-5/3}\, F(\chi). \tag{47}$$

From this connection it follows that time invariance of the asymptotic $\Lambda_\alpha\, k^\alpha$ implies the invariance of integral $\mathcal{I}_\alpha$ (Davidson, 2004). Due to analyticity arguments [the analyticity is controlled by the parameter α; see Eq. (36)] the most relevant for study are the choices $\alpha = 2,\ 4, \cdots$.

Let us here discuss briefly the sense of $\mathcal{I}_\alpha$, following Chapter 6 in the book of Davidson (2004). Kolmogorov's theory tell us only about the small scales. But the transport of momentum is usually controlled by the large eddies. The large-scale dynamics, due to Loitsyansky (1939), is understood in isotropic turbulence as starting with an integral invariant

$$I_4 = -\int d\mathbf{x}\, \langle \mathbf{u} \cdot \mathbf{u} \rangle_{(x)} \mathbf{x}^2 = 8\pi \int_0^\infty dx\, x^4 B_{LL}(x) \tag{48}$$

as a consequence of the general law of angular momentum conservation. The integral I_4 has physical significance in term of the energy spectrum rewritten to the form

$$E(k) = \frac{1}{\pi} \int_0^\infty dx\, \langle \mathbf{u} \cdot \mathbf{u} \rangle_{(x)} \sin(kx) \tag{49}$$

with the useful property: for a random array of simple eddies for fixed size l, $E(k)$ peaks at around $k \sim \pi/l$. Expanding $\sin(kx)$ into the Taylor's series and assuming that $\langle \mathbf{u} \cdot \mathbf{u} \rangle_{(x)}$ falls rapidly with x, we obtain the expansion:

$$E(k) = I_2\, k^2/4\pi^2 + I_4\, k^4/24\pi^2 + \cdots, \tag{50}$$

where $I_2 = \int d\mathbf{x} \langle \mathbf{u} \cdot \mathbf{u} \rangle_{(x)}$ is known as Saffman's invariant (Saffman, 1967a, b). Note that assuming that longitudinal correlation exponentially falls at large x, we would expect $I_2 = 0$ and $E(k) = I_4\, k^4/24\pi^2 + \cdots$. So, $I_\alpha = \mathcal{I}_\alpha$, and we have in grid turbulence two asymptotic spectra $E(k) \sim k^\alpha$, $\alpha = 2, 4$. The special cases $\alpha = 2$ and $\alpha = 4$ are studied for arbitrary dimension $d \in (2, 3)$ in detail in Section 5.1. and 5.2., respectively.

As it has been mentioned in previous paper of Adzhemyan *et al.* (1998) this asymptotic behaviour is connected with the question about the initial condition of decay. Since the self-similarity makes the time and wave-number asymptotics interconnected, the initial condition $\lim_{t\to 0} E(k, t) = \Lambda_\alpha k^\alpha$ stems simply from the asymptotic constraint (36). From that it follows that the inertial range is not formed for $t \to 0$ [since $l(t) \to 0$]. When t increases, the lower bound of the inertial range moves towards the larger spatial scales and the $E(k, t)$ spectrum attains the shape typical for the three-dimensional developed turbulence.

The remarkable property of the $\Lambda_{d-1} k^{d-1}$ energy spectrum is that it describes $\delta^{(d)}$-correlated (in the real space representation) velocity fluctuations. In addition, any increase of α ($\alpha > d - 1$) in the spectrum $\Lambda_\alpha k^\alpha$ induces more pronounced pushing of the energy towards the smaller scales, which means that the spectrum mentioned should be capable to model the response of the system on the variety of the stirring regimes. In agreement with the proposed model, the presence of strong stirring near the grid is the characteristic feature of the initial stages of the past grid evolution.

4.2. The Parametrization at the Small Scale Turbulence Structure

Finally, we give an important suggestion for the improvement of the $F(\chi)$ parametrization at large $\chi = kl$. This suggestion is based on Eq. (30). To achieve a more profound information about F we have considered the asymptotic form

$$F(\chi) \simeq 1 - h\, \chi^{\alpha_h} \qquad \text{as} \qquad \chi \gg 1 \qquad \text{and} \qquad \alpha_h < 0 \tag{51}$$

compatible with the normalization $F(\infty) = 1$. In further we describe how the unknown parameters h and α_h can be determined. From Eq. (14) we obtain

$$\psi^{\{F\}}(\chi)\Big|_{F(\chi)\to 1 - h\,\chi^{\alpha_h}} = -h\, \psi_0(d, \alpha_h)\, \chi^{\alpha_h} + \mathcal{O}(\chi^{2\alpha_h}), \tag{52}$$

where

$$\psi_0(d, \alpha_h) = c'_T \int_\triangle dq\, dp\; R_0(q, p), \tag{53}$$

$$R_0(q, p) = K(p, q)\{(q^{\alpha_h} + p^{\alpha_h})(p\, q)^{-\frac{2}{3}-d}\,[\,Q(p, q) + Q(q, p)\,] \\ -\;(1 + p^{\alpha_h})p^{-\frac{2}{3}-d} Q(p, q) - (1 + q^{\alpha_h})q^{-\frac{2}{3}-d} Q(q, p)\}.$$

Due to Eq. (28) the zeroth order term $\psi^{\{F\}}(\chi)|_{F=1}$ is not present in the series (52). Insertion of Eq. (51) into left hand side of Eq. (26) gives

$$\hat{L}(1 - h\,\chi^{\alpha_h}) = c_1\,\chi^{-2/3} - c_1\,h\;(1 + c_2\,\alpha_h\,)\;\chi^{\alpha_h-\frac{2}{3}}. \tag{54}$$

For $\chi \gg 1$ Eq. (26) acquires the form $-h\psi_0(d, \alpha_h)\chi^{\alpha_h} = c_1\chi^{-2/3}$. Its solution is

$$h = -\frac{c_1}{\psi_0\,(d, -2/3)} = \frac{c_h}{\sqrt{C_k}}, \qquad c_h = \frac{c_{10}}{\psi_0\,(d, -2/3)}, \qquad \alpha_h = -\frac{2}{3}. \tag{55}$$

5. NUMERICAL CALCULATION OF THE SCALING FUNCTION $F(\chi)$

At this stage the formulation of the problem of finding F is mathematically completed. Nevertheless, the structure of Eqs. (26), (35), (42) and (43) is out of hand of the standard approaches due to non-local character of the integral equation. Direct application of the standard numerical approaches seems to be also impossible.

Our robust solving procedure is realized by using the least–squares criterion. The subject of numerical minimization is the functional

$$\mathcal{F}\{\text{variational parameters}\} \equiv \sum_{\chi\,\in\,\text{mesh points}} \chi^{-2}\left[\hat{L}\,F(\chi) - \psi^{\{F\}}(\chi)\right]^2, \tag{56}$$

which sums up weighted squared differences $\hat{L}\,F(\chi) - \psi^{\{F\}}(\chi)$ calculated for the system of the mesh points. The variational parameters of functional $\mathcal{F}$ are included into parametrization of F [see Eq. (59) below] suggested with respect to the asymptotical requirements $F(\infty) = 1$ and (37).

The form of the functional $\mathcal{F}$ stems simply from a more general functional $\int dk\,(\partial_t E - T)^2$, which attains the minimum at $\partial_t E = T$. To find the minimum numerically steepest descent minimization procedure has been utilized. Of course, the minimization of $\mathcal{F}$ must take into account Eqs. (35), (42) and (43). Thus, the basic minimization algorithm has been combined with seldom iterative steps taking into account these equations [the impact of the additional conditions will be discussed in Section 5.1.]. The convergence towards the minimum was unviolated thanks to the weak sensitivity of $I^{\{F\}}$, C_k and c_1 upon the changes in variational parameters.

By Adzhemyan *et al.* (1998) the evolution of decay and scaling behavior has been examined in the special case $d = 3$ and $\alpha = 2$, when the Eqs. (39) give exponents (dependent only on α, see Section 4.1.)

$$\alpha_l = \frac{2}{5}, \qquad \alpha_D = -\frac{11}{5}, \qquad c_2 = -\frac{3}{11} \tag{57}$$

in compliance with the earlier result of Saffman (1967a, b) coinciding with the experimental finding [see e.g. McComb (1990)]. Using Eqs. (12) and (14) they have calculated $(c_E)_{d=3} = 0.162329$ and $(c_T)_{d=3} = 6.6\bar{6}$, and, from Eqs. (42), (43), (52), (55) and (57) they have obtained

$$c_{10} = 0.007994, \quad \psi_0\,(d = 3, -2/3) = 0.009904, \quad c_h = 0.8071, \quad C_k = \frac{1.5}{I^{\{F\}}}. \tag{58}$$

In general d-dimensional case one must have calculated value of ψ_0, which is needed for calculation of the internal parameter h defining the asymptotic behavior of the scaling function $F(\chi)$, in the whole dimension range from $d = 2$ up to $d = 3$, see Eqs. (51) and (55). The value of h can be calculated only numerically if all parameters of $F(\chi)$ are known, such that their calculation must be performed in a cycle with calculation of very function $F(\chi)$ (their parameters). Note that a shape of ψ_0 is a result of asymptotic condition for large argument χ and its value does not depend on α. The calculated dimension dependent values of ψ_0 are given in Table I for dimension from 2 up to 3:

Taking Eq. (51) into account the following improved parametrization of $F(\chi)$ has been suggested by Adzhemyan *et al.* (1998):

$$F_{[a,m,c;h]}(\chi) = \chi^{11/3}\left(\chi^4 + 2m^2\chi^2 + a^4\right)^{-11/12}\left(1 + h\,(c^2 + \chi^2)^{-1/3}\right)^{-1}. \tag{59}$$

It contains the variational parameters m, a and c and also connected parameter h. The parameters a and m determine the shape of F at small χ and its asymmetry, respectively. Note that the parameter c guarantees the analyticity of F at small χ. All of them depend on the dimension d.

Table I. The Value of $\psi_0(d)$ for Dimensions $d \in (2, 3)$

d	2	2.1	2.2	2.3	2.4
ψ_0	−0.001370	0.000927	0.002991	0.004784	0.006291
d	2.44	2.45	2.5	2.6	2.64
ψ_0	0.006557	0.006936	0.007511	0.008456	0.008758
d	2.65	2.7	2.8	2.9	3
ψ_0	0.008828	0.009144	0.009599	0.009847	0.009904

Table II. Parameters of F and the Kolmogorov Constant for $\alpha = 2$

d	a	m	c	h
3.0	1.58 ± 0.01	0.68 ± 0.02	2.91 ± 0.04	0.6415580
2.9	1.72 ± 0.01	0.70 ± 0.10	3.0 ± 0.1	0.6833563
2.8	1.92 ± 0.01	0.80 ± 0.04	3.1 ± 0.1	0.7300551
2.7	2.19 ± 0.01	0.77 ± 0.02	3.03 ± 0.05	0.7919855
2.6	2.57 ± 0.05	0.75 ± 0.05	3.08 ± 0.05	0.8708454
2.5	3.05 ± 0.05	0.77 ± 0.03	2.8 ± 0.1	0.9971045
2.4	3.82 ± 0.01	0.70 ± 0.10	3.1 ± 0.1	1.1325046
2.3	5.30 ± 0.03	0.50 ± 0.01	2.8 ± 0.1	1.3973645
2.2	8.5 ± 0.1	0.50 ± 0.01	2.8 ± 0.1	1.9510900

5.1. Parametrization of the Scaling Function for $\alpha = 2$

Minimization of the functional (56) has been performed with the resolution of fifty mesh points uniformly distributed within the interval $0 < \chi \leq 14$ and it leads to the numerical values written in following Table II:

Fruitfulness of the minimization is demonstrated in Fig. 1 where solid lines show the left hand (differential) side of Eq. (26) and circles represent discrete values of the right hand (integral) side in points of calculation of the integral ψ. Good agreement of the both sides has been achieved for dimension $d \in (2.2, 3)$. Noticeable discrepancy for small values of χ is seen for $d = 2.2$, and for $d = 2.1$, it was already impossible to find minimum of the functional $\mathcal{F}$ (in the region of $b \leq 5000$, $m \leq 1000$, $c \leq 1000$ which were the upper starting values). In other words, under the value of approximately $d = 2.2$ the solution of Eq. (26) can not be obtained.

Behavior of the transfer $T(k)$ will be discussed in Section 6. Here we limit ourselves to several remarks about transfer dependence on dimension d. It is seen from Fig. 1 that the minimum of $T(k)$ (the maximal energy flux) moves along to larger $\chi = kl$ (i.e. to vortex of a lower energy) if d decreases. Therefore, the energy fluxes are smaller for a smaller d, what means a lower absolute value of minimum on graphs. Also the width of the peak increases, i.e. the energetic region increases and the inertial interval shifts to larger χ.

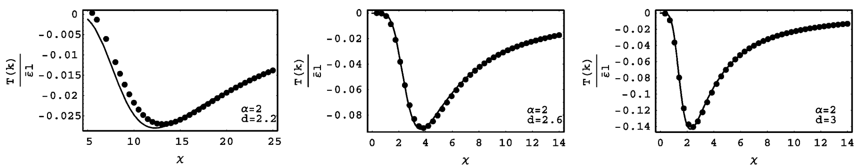

Fig. 1. Result of the minimization for $d = 2.2$, 2.6 and 3.0.

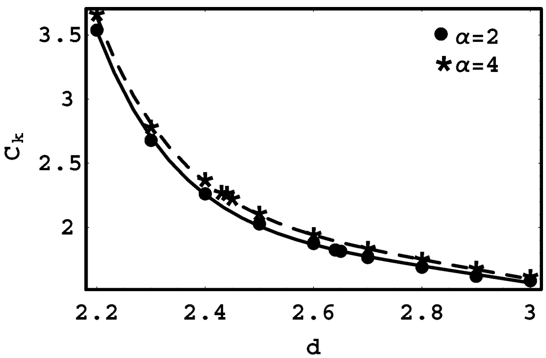

Fig. 2. The Kolmogorov constant for $\alpha = 2$ and 4.

Let us designate in our model by d_c the boundary dimension (the borderline) whereunder the function $T(k)$ starts to acquire a positive value. The value of $d_c = 2.65$ was obtained for $\alpha = 2$. Note that the energy transport is normalized in order that it is negative for the energy flux directed from large scales to small scales. Consequently the region of the inverse energy cascade is the region of positive energy flux. Calculation shows that for d decreasing to 2.1 the region of inverse energy flux flues to wider interval of χ, and deeper to positive region of energy transfer (see detail analysis of $T(k)$ below). One can be convinced that the differential part of Eq. (26) is also positive in case of $\alpha = 2$, and the critical d_c belongs to the interval (2.4, 2.5).

Dependence of the Kolmogorov constant $\mathcal{C}_k$ on dimension d demonstrates Fig. 2 (the solid line). Our minimalization procedure shows that $\mathcal{C}_k$ arises for decreasing value of d and it diverges for $d < 2.2$ in correspondence with the result of Adzhmyan *et al.* (1998) where authors found $\mathcal{C}_k \to \infty$ for $d \to 2.066$ for model with homogeneous turbulence only in the inertial interval.

A suitable step to compare the theory and experiment is to transform $E(k, t)$ into the form of the longitudinal spectrum $E_{\|}$, [see e.g. Monin and Yaglom (1975)]. This transformation is realized with help of Eq. (40) with the result:

$$E_{\|}\left(\frac{\chi}{l(t)}, t\right) = \int_1^{\infty} dz \left(z^2 - 1\right) z^{-3} E(kz, t)$$
$$= [u(t)]^2 \, l(t) \, \mathcal{C}_k \, \chi^{-5/3} \int_0^1 dz \, (1 - z^2) \, z^{2/3} \, F\left(\frac{\chi}{z}\right). \qquad (60)$$

The Fig. 3 shows the normalized spectrum $E_{||}/u^2 l$ in dependence on $\chi \equiv kl$ for dimensions $d = 2.2,\ 2.3,\ 2.6,\ 3$. One can see that the inertial interval moves to larger value of χ for decreasing d down to 2.2, and absolute value of energy spectrum decreases for decreasing d.

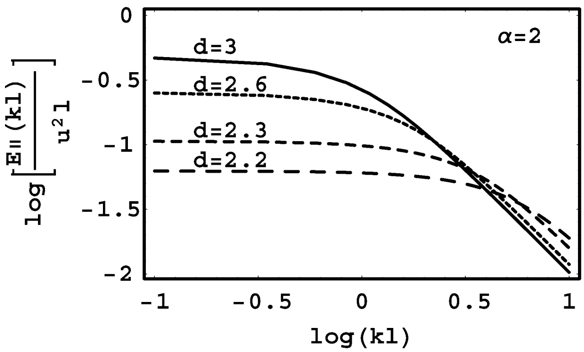

Fig. 3. Longitudinal component of the energy spectrum scaled to $u^2 l$ in logarithmic coordinates depending on $\chi = kl$ for four values of d ($d = 2.2$, $d = 2.3$, $d = 2.6$ and $d = 3$) and $\alpha = 2$.

5.2. Parametrization of the Scaling Function for $\alpha = 4$

Similar results of examination of the evolution of decay and scaling behavior have been obtained also in the case of $\alpha = 4$, where the Eqs. (39) give exponents

$$\alpha_l = \frac{2}{7}, \qquad \alpha_D = -\frac{17}{7}, \qquad c_2 = -\frac{3}{17} \tag{61}$$

Minimizing of the functional (56) with the resolution of fifty mesh points uniformly distributed within the interval $0 < \chi \leq 14$ leads to the following numerical values in Table III:

Results of the minimization are shown in Fig. 4. The character of behaviour of the spectrum $T(k)$ is very similar to the case of $\alpha = 2$. The discrepancy of both the l.h.s. and r.h.s. of Eq. (26) becomes to be notable for values $d < 2.2$, i.e. for d rather smaller then in the case of $\alpha = 2$. Comparison of the energy transfer $T(k)$

Table III. Parameters of F and the Kolmogorov Constant for $\alpha = 4$

d	a	m	c	h
3.0	0.85 ± 0.01	0.95 ± 0.01	2.84 ± 0.02	0.5897589
2.9	1.00 ± 0.05	1.00 ± 0.05	2.95 ± 0.05	0.6836230
2.8	1.14 ± 0.02	1.1 ± 0.1	2.9 ± 0.1	0.7296646
2.7	1.40 ± 0.02	1.17 ± 0.03	3.0 ± 0.1	0.7857056
2.6	1.83 ± 0.01	1.15 ± 0.05	2.7 ± 0.1	0.8560200
2.5	2.01 ± 0.01	1.50 ± 0.02	2.5 ± 0.1	0.9634254
2.4	2.8 ± 0.1	1.7 ± 0.1	3.0 ± 0.1	1.1084236
2.3	3.8 ± 0.1	2.2 ± 0.1	2.6 ± 0.1	1.3626610
2.2	7.2 ± 0.1	1.9 ± 0.1	2.8 ± 0.1	1.8811520

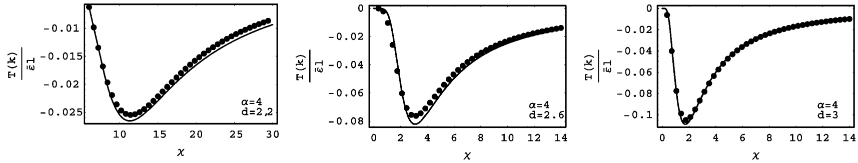

Fig. 4. Result of the minimization for $d = 2.2$, 2.6 and 3.0.

in the cases of $\alpha = 2$ and $\alpha = 4$ for fixed dimension d shows that the energy flux is smaller for $\alpha = 4$. The borderline for $T(k)$ is equal $d_c = 2.45$, but unlike the case of $\alpha = 2$, here the differential part of Eq. (26) has no borderline because it is negative in the whole interval of d.

The Kolmogorov constant acquires slightly higher value then in the case $\alpha = 2$ in the whole interval of d, see Fig. 2, and here it diverges for rather lower dimension ($d \doteq 2.1$). The normalized longitudinal spectrum is shown in Fig. 5. Comparison of this figure with Fig. 3 shows that absolute values of the spectra for $\alpha = 4$ acquire slightly lower values in comparison with the spectra for $\alpha = 2$.

6. ANALYSIS OF THE ENERGY TRANSFER

The energy transfer $T(k, t)$, see Eq. (7), is in close relation with triple velocity correlation function

$$\langle v_i(\mathbf{k})\, v_j(\mathbf{p})\, v_l(\mathbf{q})\rangle \;\equiv\; \delta(\mathbf{k}+\mathbf{p}+\mathbf{q})\; D_{ijl}(\mathbf{kpq}) \qquad (62)$$

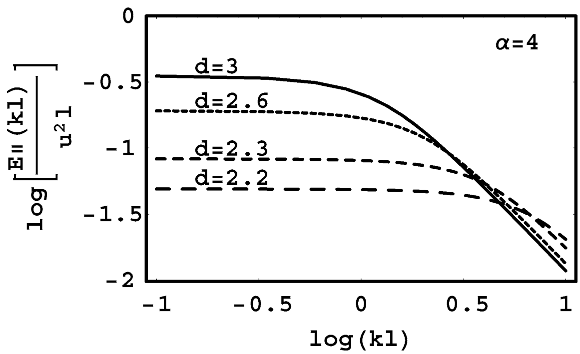

Fig. 5. Longitudinal component of the energy spectrum scaled to $u^2 l$ in logarithmic coordinates depending on $\chi = kl$ for four values of d ($d = 2.2$, $d = 2.3$, $d = 2.6$ and $d = 3$) and $\alpha = 4$.

which determines the dynamics of the decay of turbulence. From the integral identity $\int_0^\infty dk\, T(k) = 0$ one has $T(k) = -(2\pi)^d\, \partial_{k_i}\, J_i(\mathbf{k})$, where $J_i(\mathbf{k})$ represents the spectral density of energy flux through the spectrum. Owing to isotropy one can write

$$J_i(\mathbf{k}) = k_i\, J(k), \qquad I(k) = S_d\, J(k)\, k^d, \tag{63}$$

where $I(k)$ has the meaning of the total energy flux outwards through the surface of a sphere of radius k in momentum space. The flux $I(k)$ has to satisfy the obvious limiting $I(0) = I(\infty) = 0$. In the inertial range both the viscosity and the energy injection are unimportant, so there $T(k) = 0$. Due to zero of the total integral of $T(k)$ in this case the direction of the energy flux must differ in different regions of spectrum.

More complex picture of the energy transfer along the whole spectrum can be given by detailed analysis of the integrand $R(q, p; \chi)$ in (15). The negative energy transfer corresponds to "normal" energy cascade when the energy flows from large to small scales (to large $\mathbf{k}$). The positive energy transfer corresponds to an inverse energy cascade with opposite energy flow direction. In 3-dimensional developed turbulence the *full* energy transfer is always negative in correspondence with phenomenological image of the vortex cascade decay (McComb, 1990; Davidson, 2004). Separate contributions into the transfer in a region of definition of wave vector "triade" $\{\mathbf{kpq}\}$, fulfilling the condition $\mathbf{k} + \mathbf{p} + \mathbf{q} = 0$, can possess different signs. Positive contributions start to increase in the case of decreasing dimension d, what leads to the inverse cascade of energy (see below).

The integrand $R(q, p; \chi)$ possesses a nontrivial topology and is symmetric to the axis $p = q$ as it follows from (15). In general, three regions of the "triade" $\{\mathbf{kpq}\}$ in the integration range Ω can be distinguished: two with creation of the inverse energy flow ($R > 0$) and one with the direct energy flow ($R < 0$).

The first region (I.) is a region of "local" triades (all wave vectors of approximately the same magnitude) where an inverse energy flow is generated. In the dimensionless variables it extends to the region of small $\{p, q\}$. Value of R decreases towards to non-local triades and it passes into another region.

In *the second region* (II.) of the inverse energy flow which ranges along both borders of the integration region ($p = q - 1, p = q + 1$). This region corresponds to "strongly non-local" triades when $q \ncong p$ and $p, q \gg k$. This region appears for sufficiently high values of p, q. Here the integrand gradually achieves its maximum and towards to more non-local triades it slowly decreases to zero .

The third region (III.) covers bulk of the integration range along the axis of symmetry of R where it decreases to negative values with an increase of non-locality to some minimum and then gradually increases to zero. These "non-local" triades (but $p \cong q$) generate the direct energy flux.

Note that if the wave vectors $\{k, p, q\}$ are transformed back to dimensional vectors in units of $1/l$, see Eq. (20), then these triades can be figured out by

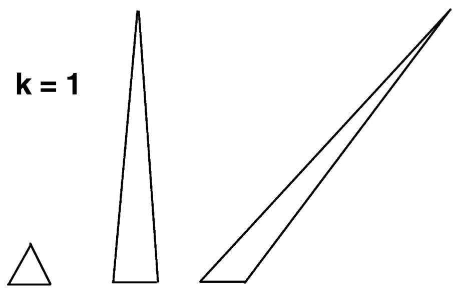

Fig. 6. Example of local, non-local and strongly non-local triade (from the left to the right).

the schematic triangles. The example of local (I.), non-local (III.) and strongly non-local (II.) triades is shown in Fig. 6.

Ranges of the three regions as well as its contribution into integral of the energy transfer nontrivially vary with changes of $\chi = kl$. Below we shortly analyze the triade contribution for $\alpha = 2$ as an example.

6.1. Analysis of Energy Transfer for $d = 3$

In this section we consider as an example the contribution of different "triades" $\{\mathbf{kpq}\}$ in the case of $d = 3$, $\alpha = 2$ and the parametrization of the scale function $F(\chi)$ by the following numerical values: $b = 1.570, m = 0.657, c = 2.838, h = 0.643$. The sign of the contribution is demonstrated in Fig. (7a–d) where the integrand $R(q, p; \chi)$ is presented for values of $\chi = 0.1, 0.9, 1.5$ and 3. Here the surface out of the integration range Ω is not displayed.

One can see that a positive contribution from the I. region of local triades for $\chi = 0.1$, see Fig. 7a, is negligible and it starts to increase for increasing value of χ. The increasing positive lobe can be observed also for $p \cong q \cong 1$.

Notable negative contribution is given by the non-local triades in the III. region. The II. strong non-local region with a positive contribution is very significant. Here the integrand R acquires considerably lower absolute values in comparison with values in the III. region, and as well R decreases essentially slower here then R arises in the III. region. In result, the values in the both regions compensate each other and give very small but negative value of R.

As χ increases the relative contribution of various regions into the energy transfer varies. One can see in Fig. 7b that for $\chi = 0.9$ the positive contribution of local triades (I. region) starts to arise. Nevertheless, this contribution together with positive values in other regions can not compensate rising contribution of the central region of non-local triades, so the integral transfer acquires gradually larger negative values.

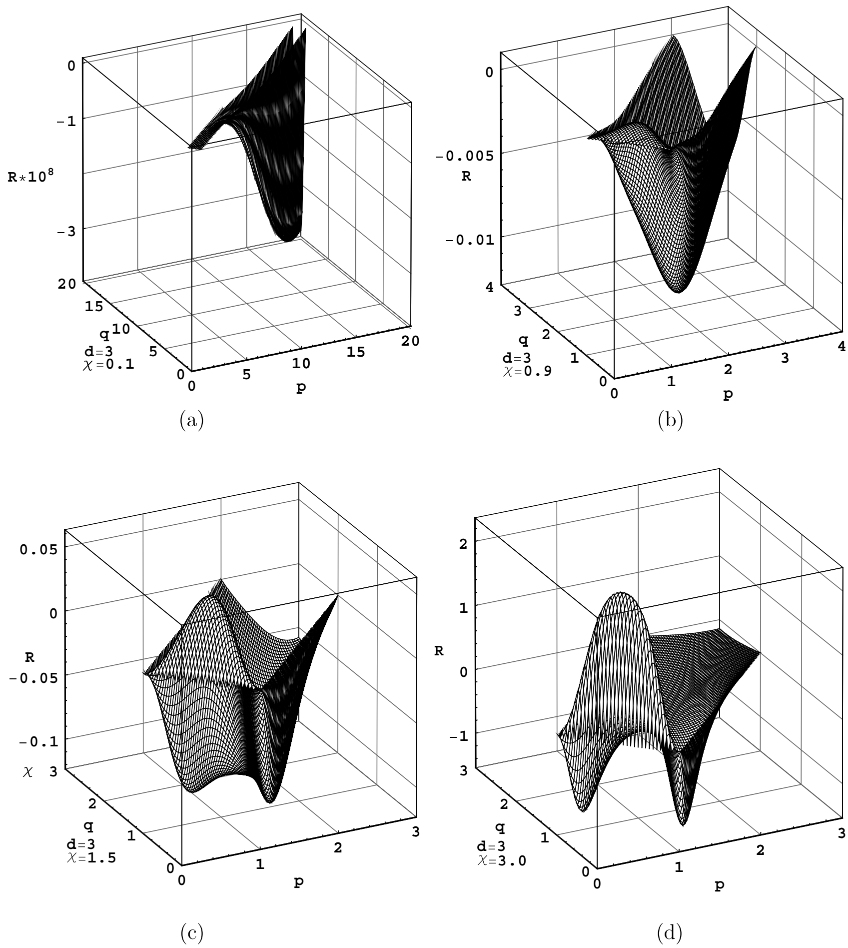

Fig. 7. Transfer for $d = 3$ and $\chi = 0.1, 0.9$ (at the top), $\chi = 1.5, 3$ (at the bottom).

Positive contribution of the local triades starts to arise when χ increases, and the importance of positive contribution of the strong non-local triades starts to be negligible. It is demonstrated in Fig. 7c and d showing the value of R for $\chi = 1.5$ and $\chi = 3.0$, respectively. One can see changes in the region of non-local triades with a negative contribution, where two symmetrical lobes arise with minimal values. Positive contribution of the local triades starts to predominate over negative contribution of the non-local triades and so the energy transfer starts to arise.

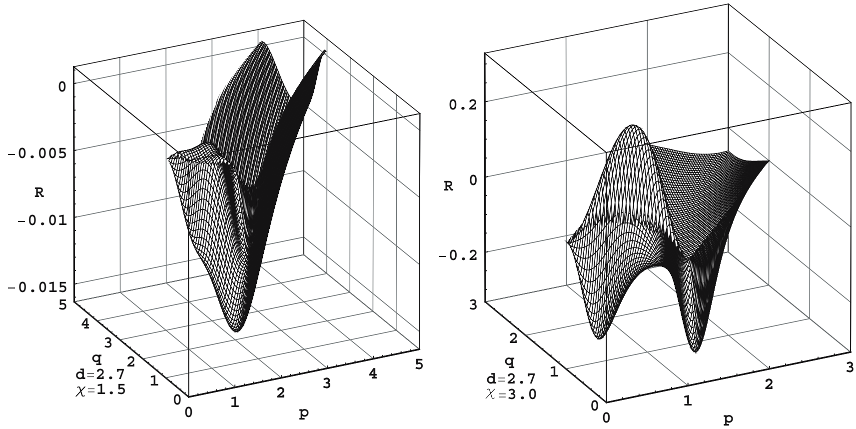

Fig. 8. Transfer for $d = 2.7$, $\chi = 1.5, 3$.

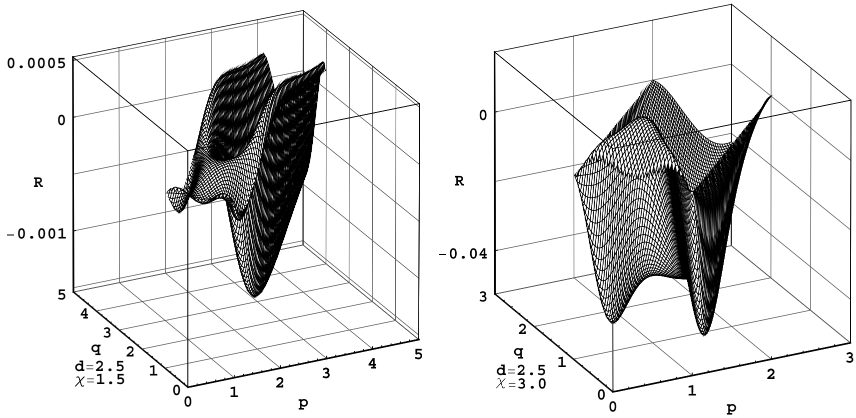

Fig. 9. Transfer for $d = 2.5$, $\chi = 1.5, 3$.

Analysis of the integrand R for a larger value of $\chi \gtrsim 5$ shows that two positive local lobes in the I. region of local triades start to expand and the integral energy transfer gradually increases from its negative value towards to zero.

6.2. Analysis of Energy Transfer for $d < 3$

Parametrization of the scaling function $F(\chi)$ for dimensions $d \leq 3$ and $\alpha = 2$ is given in Table II. It is well known that full integral of the energy transfer can

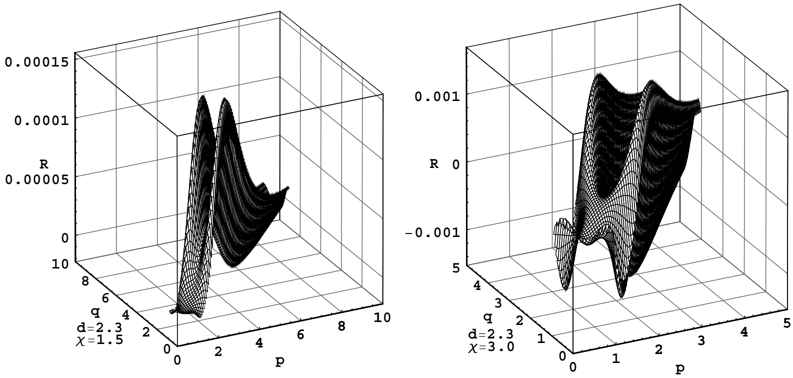

Fig. 10. Transfer for $d = 2.3$, $\chi = 1.5, 3$.

be positive for $d = 2$, i.e. the energy transfer can be inverse (McComb, 1990; Davidson, 2004). The negative energy transfer (for $d = 3$) gradually changes to the inverse positive energy transfer as it demonstrate the Figs. 8–10 in two cases of $\chi = 1.5$ and 3. The integrand $R(\chi)$ possesses rather complicated structure in the case of $d < 3$ in comparison with the case of $d = 3$. Another regions of both the direct as well as the inverse energy flows arise in the integration limits. It is important that for $d < 3$ and small χ positive contribution predominates over negative ones, and the full integral $T(k)$, (14, 15) can be positive.

7. CONCLUSION

The model is presented which enables to calculate the scaling forms of the energy spectrum and the energy transfer of a decaying turbulence in the d-dimensional case for $d \in (2, 3)$. The aim of the work was to study the isotropic and homogeneous turbulence of the energy–containing and adjacent inertial range in which the decay laws acquire the power form. The initial point of our study represents the results obtained for the stationary model of the randomly forced turbulence with the extended form of the forcing. To describe a free decay of the past–grid turbulence the results of the stationary model of the randomly stirred fluid were modified. The assumption about the slowness of the time evolution takes place for the high k and late time stages of the power decay. But for them, in addition, the Reynolds number must be kept sufficiently large.

Since the spectrum behavior in the interval of small k in the Navier–Stokes turbulence has not been firmly established up to now, we have adopted for this aim the Saffman and Loitsyansky hypothesis known for $d = 3$. The computation of the scaling function has been performed for $d \in (2, 3)$ and $\alpha = 2$ and $\alpha = 4$. In the both cases the evaluated longitudinal energy spectrum $E_{\|}$ exhibits a promising agreement with the turbulence data in the wave–number range from $0.1/l$ up to $10/l$. In this range $E_{\|}$ changes approximately by two orders of magnitude. By Davidson (2004), while a stationary grid may well give an $E \sim k^4$ spectrum, as suggested by the $-5/2$ decay law in the final period of decay, a grid which is vigorously shaken might be able to impart sufficienty momentum to the turbulence to ensure an $E \sim k^2$ spectrum. (In such a case $\mathcal{I}_2$ will not exist.) In d-dimensions the situation can be more complex. The entire issue of $E \sim k^4$ versus $E \sim k^2$ spectra is still a matter of some controversy.

The shape of the function $T(\chi/l, t)$ is in a qualitative compliance with the canonical expectations for the energy–containing range (McComb, 1990). Nevertheless, due to insufficient accuracy and reproducibility of the available experimental data, the quantitative testing of the third order statistics represents a doubtful task.

ACKNOWLEDGEMENT

This work was supported by Science and Technology Assistance Agency under the contract No. APVT-51-027904, and by SAS, project No. 2/6193.

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